**Variational Technique**

Another way to approximately solve ODE’s is to create an ‘action’ functional of the dependent variable, y. Minimizing this action should yield the ODE equation. But instead of solving the ODE to minimize the action, we approximately minimize the action to approximately solve the ODE.

**Action for Sturm-Louiville ODE’s**

See the ODE pdf for much more on this. So consider action:



(can think of as sort of like harmonic oscillator Hamiltonian) Then Lagrange equations of motion are:



and so we come to:



This is of course the Sturm-Louiville ODE. So a function which approximately solves this ODE (and its linear/homogeneous boundary conditions) would be one which approximately minimizes (subject to satisfaction of boundary conditions) the corresponding action. Now we can say that if we had a general differential equation:



An appropriate S would be:



where the weight function is the familiar:



**Example**

Let’s consider the Sturm-Louiville action with p = -1, q = x4.



Let’s examine functions of the form,



Then,



Want to extremize this. So,



**Eigenvalue Equations and the Action**

Eigenvalue equations can be thought of as minimizing a Sturm-Liouville action subject to the additional constraint of normalization of y2 w/r to a weight function, ρ. We would minimize the action,



where I've chosen the Lagrange multiplier to be λ/2 for convenience. Now it will be the case that S has a relative minimum only for special values of λ - the eigenvalues. And these minima are the eigenfunctions, u. The Euler-Lagrange equation would come to:



Now consider a related action, S´.



Now consider that we plug a normalized function into S´. If S´ is minimized, the result will be λ. If we look for the absolute minima of S´, then we'll get the first eigenfunction u1 , and its eigenvalue λ1. If we look for the minima next up, we get u2 and λ2. A convenient mathematical way of obtaining the next eigenvector is to minimize S with respect to a normalized y, as so far, but also w/r to one that is orthogonal to u1. So now there are two constraints.



And using Lagrange's method we would seek to minimize:



So if we obtain the absolute minima of the new S´´ we get u2 , and λ2, and so on. Alternatively we could use normalized test functions that are constructed to be orthogonal to u1. Specifically we would have:



and,



and so on (where the 'dot' product is really the integral - dot product. But as it turns out we can even say that:



and,



etc. where v1, v2, etc. are arbitrary functions. So the eigenvalue λn is the *maximum* minima of S over any space excluding n-1 arbitrary functions. A neat application of this result is that the function q(x) doesn't affect the asymptotic behavior of the eigenvalues - as we could expect from the born approximation. Consider:



The action is:



The maximum minima of A, will be less than or equal to the maximum minima of:



if it is the case that r is always non-negative. But these eigenvalues are λn(0) = (nπ)2. Now since the actual eigenvalues λn go to infinity as n goes to infinity, it is the case that asympotically λn ~ (nπ)2.

**Miscellaneous Properties**

Using the framework above, we can show that the GS can have no nodes between its boundary points. And the nth order wavefunction must have nodes therefore in order for it to be orthogonal to the GS. However, it may have no more than n nodes all together.

**Appendix**

Some good old-fashioned functional calculus stuff. Variational methods come into play frequently. For example suppose we want to determine the curve connecting points (0,0) and (1,1), that minimized arc length. Then we write an expression for arc length (in Cartesian coordinates I suppose).



Then to determine what curve, y, minimizes S, subject to the boundary conditions, we take a functional derivative,



and so we have:



In general, if we have an Action of the form,



Then our equation would read



This is perhaps more enlightening when derived the following way, which makes clear that we're setting the variation at the end points to zero. Which means that the behavior at the end points is fixed - clamped.



Now consider something else. Suppose we want to maximize our action subject to a constraint on the actual form of y . Then we could use the method of Lagrange multipliers. So suppose our constraint takes the form:



Then we would minimize:



To come to the equation:

